# Math 325 Probability Theory

2/2/23

$X$ discrete r.v.

$$\mathbb{E}(X) = \sum_x x\,\mathbb{P}(X = x)$$

$$= \sum_x x\, p_X(x)$$

$$g : \mathbb{R} \to \mathbb{R}$$

$$\mathbb{E}(g(X)) = \sum_x g(x)\, p_X(x)$$

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$$V(X) = \mathbb{E}\left((X-\mu)^2\right) = \sigma^2$$

$$\mu = \mathbb{E}(X)$$

$$V(X) = \sum_x P_X(x)(x-\mu)^2$$

$$V(X) = \mathbb{E}(X^2 - 2\mu X + \mu^2) =$$

$$= \mathbb{E}(X^2) - 2\mu \mathbb{E}(X) + \mu^2$$

$$= \mathbb{E}(X^2) - 2\mu^2 + \mu^2 =$$

$$= \mathbb{E}(X^2) - \mathbb{E}(X)^2$$

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$$var(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$$

$$var(X) \geq 0$$

$$\mathbb{E}(X^2) \geq \mathbb{E}(X)^2$$

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$X$ Bernoulli

$$\mathbb{E}(X^2) = p = 1^2 \cdot p + 0^2 \cdot (1-p)$$

$$\operatorname{var} X = p - p^2 = p(1-p)$$

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$Y$

$Y = a$ prob $p$

$Y = b$ prob $(1-p)$

$X$ Bernoulli prob $p$

$Y = b + (a-b)X$ or

define

$$X = \frac{Y-b}{a-b} \implies X \text{ is Bernoulli}$$

$$\mathbb{E}(Y) = \mathbb{E}(b + (a-b)X) =$$

$$= b + (a-b)\,\mathbb{E}(X) =$$

$$= b + (a-b)\,p =$$

$$= ap + b(1-p)$$

$X$ is a r.v. with $\mathbb{E}(X) = 0$

$$\mathrm{Var}(aX) = \mathbb{E}\left((aX)^2\right) =$$
$$= \mathbb{E}(a^2 X^2) = a^2 \mathbb{E}(X^2)$$
$$= a^2 \mathrm{Var}(X)$$

$$\mathbb{E}(aX) = a\,\mathbb{E}(X) = 0$$

$$\mathrm{var}(aX) = \mathbb{E}\left((aX)^2\right) - \mathbb{E}(aX)^2 =$$
$$= a^2 \mathbb{E}(X^2) >$$
$$= a^2 \mathbb{E}(X^2) - a^2 \mathbb{E}(X)^2 =$$
$$= a^2 \mathrm{var}(X)$$

$$\sigma = \sqrt{\sigma^2}$$

$$\sigma_{aX} = |a|\,\sigma_X$$

Suppose $X$ is a r.v.

with variance $\sigma^2$

$$\operatorname{var}(X - 1) = \sigma^2$$

$$\operatorname{var}(X - a) = \mathbb{E}\left(\left[(X-a) - \mathbb{E}(X-a)\right]^2\right) =$$

$$\mathbb{E}(X - a) = \mathbb{E}(X) - a$$

$$= \mathbb{E}\left(\left(X - \mathbb{E}(X)\right)^2\right) = \operatorname{var}(X)$$

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population.

100 ind

99 10 K

1 1000K

$x_i \qquad i = 1 \dots N \qquad x_i \le x_{i+1}$

$$\bar{x} = \frac{1}{N} \sum_i x_i$$

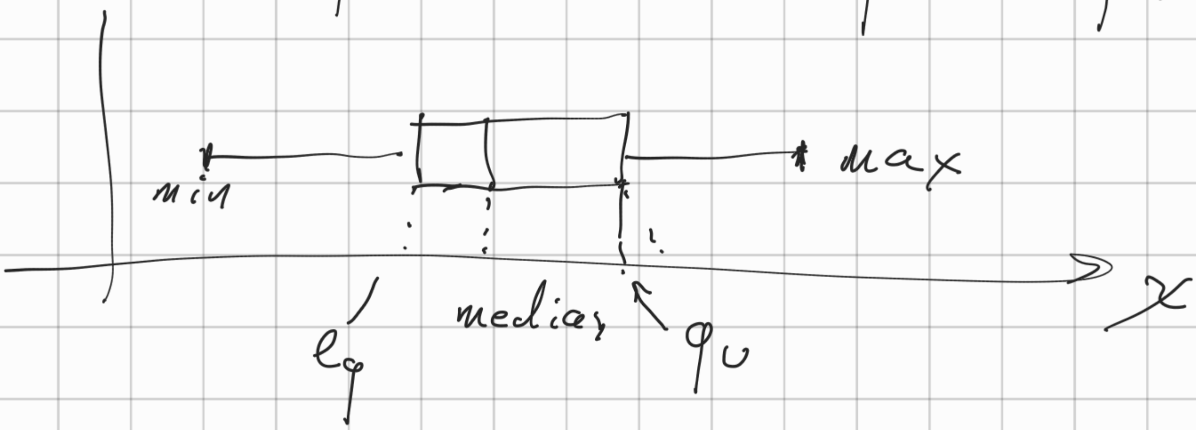
median

$$\tilde{x} = \begin{cases} x_{\frac{N+1}{2}} & \text{if } N \text{ is odd} \\ \frac{1}{2}\left(x_{\frac{N}{2}} + x_{\frac{N}{2}+1}\right) & \text{if } N \text{ is even} \end{cases}$$

$x_0 \; \dots \; x_{\frac{N}{2}}$ $\quad q_l$ lower quartile

$x_{\frac{N}{2}+1} \; \dots \; x_N$ $\quad q_u$ upper quartile

4th spread $= q_u - q_l$




Box plot

$X$ is geometric

$$p_X(x) = (1-p)^{x-1} p$$

$$E(X(X-1)) =$$

$$p \sum_{x=1}^{\infty} x(x-1)(1-p)^{x-1} =$$

$$p(1-p) \sum_{x=0}^{\infty} x(x-1)(1-p)^{x-2} =$$

$$p(1-p) \frac{d^2}{dp^2} \sum_{x=0}^{\infty} (1-p)^x =$$

$$p(1-p) \frac{d^2}{dp^2} \frac{1}{p} = p(1-p) \frac{2}{p^3}$$

$$E(X(X-1)) = E(X^2) - E(X)$$

$$\operatorname{var}(X) = E(X(X-1)) + E(X) - E(X)^2$$

$$= (1-p)/p^2$$

$$\mathbb{E}(g(X)) = \sum_x g(x)\, p_X(x)$$

$X$ is geometric $\Rightarrow$

$$p_X(x) = (1-p)^{x-1} p$$

$$g(x) = x(x-1)$$

$$\mathbb{E}(X(X-1)) = \sum_x x(x-1)(1-p)^{x-1} p$$

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$X$ is Binomial $N, p$

$X = \sum_{i=1}^{N} Y_i$ with $Y_i$

Bernoulli par $p$

$$\mathrm{Var}(X) = N\, \mathrm{var}(Y_1) =$$

$$= N p (1-p)$$

$$\mathbb{E}(X^2) = \mathbb{E}\left(\left(\sum_i Y_i\right)^2\right) =$$

$$= \sum_{i,j} \mathbb{E}(Y_i Y_j)$$

$$\mathbb{E}(Y_i Y_j) = \mathbb{E}(Y_i)\mathbb{E}(Y_j).$$

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If $B_i$ are a partition

$$\mathbb{P}(A \mid B_i) = \frac{\mathbb{P}(A \cap B_i)}{\mathbb{P}(B_i)}$$

If $X$ is a discrete r.v.

$$\mathbb{P}(X = x \mid B_i) = p_X(x \mid B_i)$$

$\mathbb{P}(\cdot \mid B_i)$ is a prob.

$$\mathbb{E}(X \mid B_i) = \sum_x x \, p_X(x \mid B_i)$$

$$\mathbb{P}(A) = \sum_i \mathbb{P}(A \mid B_i)\,\mathbb{P}(B_i)$$

$$\mathbb{E}(X) = \sum_x x\,\mathbb{P}(X = x) =$$

$$\sum_x x \sum_i \mathbb{P}(X = x \mid B_i)\,\mathbb{P}(B_i)$$

$$= \sum_i \sum_x x\,\mathbb{P}(X = x \mid B_i)\;\mathbb{P}(B_i)$$

$$= \sum_i \mathbb{E}(X \mid B_i)\,\mathbb{P}(B_i)$$

If $Y$ is another discrete r.v

$B_y = \{\omega \mid Y(\omega) = y\}$ for $y \in \mathrm{Im}(Y)$

This is a partition

$$\mathbb{E}(X) = \sum_y \mathbb{E}(X \mid Y = y)\,p_Y(y)$$

Theorem: if $X$ is a r.v. and $B_i$ a partition

$$\mathbb{E}(X) = \sum_i \mathbb{E}(X \mid B_i)\,\mathbb{P}(B_i)$$

If $Y$ is another r.v.

$$\mathbb{E}(X) = \sum_y \mathbb{E}(X \mid Y = y)\, P_Y(y)$$

$\mathbb{E}(X \mid Y = y)$ is a function of $y$ That is called The conditional expectation of $X$ given $Y$.